BRANCHING OF SOLUTIONS IN THE PROBLEM OF THE WAVY FLOW OF A VISCOUS LIQUID WITH A FREE BOUNDARY

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INTRODUCTION

The wavy flow of thin layers of a viscous liquid was discussed in [1, 2]. The existence of nonlinear stationary waves at the surface of a liquid flowing down along a vertical wall was shown for the first time in [3]. One chapter of the monograph [4] is devoted to a discussion of different flow conditions in falling films.

The linear problem of the stability of plane-parallel flow with a free boundary was studied on the basis of the Navier–Stokes equations in [5-9]. The nonlinear problem was investigated in [10-14] within the framework of the equations of P. L. Kapitsa; the Korteweg–de Vries equation and equations similar to it were used in [15, 16] to describe the flow in a liquid film. Nonlinear flow conditions in a liquid film and nonlinear stability were studied in [17-20] on the basis of the Navier–Stokes equations in a long-wave approximation. In the two latter works the Landau constant was calculated using a modified Reynolds–Potter method [21] and a conclusion was drawn with respect to the absence of long-wave subcritical motions.

1. Wavy Conditions near the Threshold of Stability

We consider a layer of viscous incompressible liquid with a density ρ and a viscosity ν, flowing down under the action of the force of gravity $g = 981$ cm/sec² along a flat surface inclined to the horizontal at an angle χ. We shall take as given the mass flow rate of the liquid Γ, defined as the time-averaged value of the mass of liquid passing through a transverse cross section and referred to unit width of the layer. As the scales of length, time, and mass, respectively, we take the quantities $(\nu^2/g)^{1/3}$, $(\nu/g^2)^{1/3}$, $\rho\nu^2/g$ and introduce the dimensionless parameters

$$\mathrm{Re} = \Gamma/\nu\rho,\ \gamma = (T/\nu\rho)(\nu g)^{-1/3},$$

the first of which is the Reynolds number, based on the mass flow rate, while the second characterizes the physical properties of the liquid. In such a statement, depending on the Reynolds number, the thickness of the layer is regarded as unknown and subject to determination. We introduce the Cartesian rectangular system of coordinates $0'x'y$, locating its origin at the bottom of the channel and directing the x' axis downward along the flow and the y axis toward the free boundary. We shall be interested in those solutions of the equations of hydrodynamics periodic with respect to time and having the form of stationary waves that run along the x' axis with an unknown phase velocity c, i.e., solutions depending periodically on the time t and the coordinate $x'(x = x' - ct)$. In this case the Navier–Stokes equations are conveniently written in the form of the equations of motion of a continuous medium in the directions

$$\begin{gathered} Du = \tau - v_x,\ Dv = -u_x, \\ D\sigma = \cos\chi - \tau_x - cv_x + uv_x - vu_x, \\ D\tau = -\sin\chi - \sigma_x - 4u_{xx} - cu_x + uu_x - vv_x + v\tau, \end{gathered} \tag{1.1}$$

where $D = \partial/\partial y$; the subscript x denotes a partial derivative with respect to the argument x; u and v are the longitudinal and transverse components of the velocity vector; and σ and τ are, respectively, the normal ($\sigma = -p + 2Dv$; p is the pressure) and tangential ($\tau = Du + v_x$) stresses in the liquid film. The first equation of the system (1.1) is actually a definition of the quantity τ; the second is the equation of continuity; and the third and fourth equations express the law of conservation of momentum in projections on the y and x' axes, respectively. The solution of the system (1.1) must be $2\pi/k$-periodic (k is a given wave num-

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ber) along the coordinate x. At the boundaries of the layer of liquid y = 0 (solid wall) and $y = \zeta(x)$ (free surface) the following conditions are satisfied:

$$u = v = 0 \ (y = 0); \tag{1.2}$$

$$v - (u - c)\zeta_x = 0 \ (y = \zeta); \tag{1.3}$$

$$\sigma = -p_a + \gamma\zeta_{xx}(1 + \zeta_x^2)^{-3/2} + \tau\zeta_x \ (y = \zeta); \tag{1.4}$$

$$\tau = 4u_x\zeta_x(1 - \zeta_x^2)^{-1} \ (y = \zeta), \tag{1.5}$$

where p_a = const is the dimensionless value of the atmospheric pressure. For wavy solutions the mean with respect to time coincides with the mean with respect to the variable x; therefore, the condition for the mass flow rate in dimensionless variables assumes the form

$$\mathrm{Re} = \frac{k}{2\pi}\int\limits_0^{2\pi/k}\int\limits_0^{\zeta(x)} u(x, y)\,dy dx. \tag{1.6}$$

The system (1.1)-(1.6) admits of the known exact solution $u = u'$, $v = v'$, $\sigma = \sigma'$, $\tau = \tau'$, $\zeta = \zeta'$, described by the formulas

$$u' = \sin\chi(\mu y - 0{,}5y^2),\ v' = 0,\quad \sigma' = -p_a + \cos\chi(y - \mu),$$
$$\tau' = \sin\chi(\mu - y),\quad \zeta' = \mu,\ \mu = (3\mathrm{Re}/\sin\chi)^{1/3} \tag{1.7}$$

and, corresponding to, as can be seen, plane-parallel flow in a layer of thickness μ with a flat free boundary. The problem consists in finding a wavy flow of the liquid differing from (1.7).

Following the method for calculations of the autovibrations of continuous media proposed in [22-24], and limiting ourselves to wavy conditions of small amplitude, branching out from plane-parallel flow (1.7), we shall seek the solution of the problem posed in the form

$$\{u, v, \sigma, \tau, \zeta, c\} = \{u', v', \sigma', \tau', \zeta', c_0\} + \sum_{m=1}^{\infty}\varepsilon^m\{u_m, v_m, \sigma_m, \tau_m, \zeta_m, c_m\},$$
$$\mu = [3(\mathrm{Re}_0 + \delta\varepsilon^2)/\sin\chi]^{1/3} = \sum_{m=0}^{\infty}\mu_{2m}(\delta\varepsilon^2)^m, \tag{1.8}$$
$$\mathrm{Re} = \mathrm{Re}_0 + \delta\varepsilon^2,\ \mu_0 = (3\mathrm{Re}_0/\sin\chi)^{1/3},\ \mu_2 = (9\mathrm{Re}_0^2\sin\chi)^{-1/3},$$

where $\varepsilon > 0$ is a small parameter; Re_0, c_0, and μ_0 are the critical values of the Reynolds number, the phase velocity, and the thickness of the layer, determined in accordance with the linear theory; and the value of δ, equal to +1 or −1, is responsible for the sign of the increment of the Reynolds number. The value of the latter is previously unknown and is determined during the course of the solution of the problem.

We carry the boundary conditions at $y = \zeta(x)$ to the unperturbed boundary $y = \mu_0$, expanding all the functions of the coordinate y entering into (1.3)-(1.5) in Taylor series in the neighborhood of the point $y = \mu_0$. We then substitute the expansions of (1.8) into Eqs. (1.1)-(1.6) and collect terms with identical powers of the parameter ε. As a result, we arrive at a series of recurrent linear problems (m = 1, 2, 3, ...)

$$Du_m = \tau_m - v_{mx},\ Dv_m = -u_{mx},\quad D\sigma_m = (U - c_0)v_{mx} - \tau_{mx} + F_m,$$
$$D\tau_m = (U - c_0)u_{mx} + \mathrm{DU}v_m - 4u_{mxx} - \sigma_{mx} + G_m,$$
$$U = \sin\chi(\mu_0 y - y^2/2),\ u_m = v_m = 0\ (y = 0), \tag{1.9}$$
$$v_m - V\zeta_{mx} = K_m(y = \mu_0),\ V = \frac{1}{2}\mu_0^2\sin\chi - c_0,$$
$$\sigma_m + \zeta_m\cos\chi - \gamma\zeta_{mxx} = L_m(y = \mu_0),\quad \tau_m - \zeta_m\sin\chi = S_m(y = \mu_0),$$

for which it is required to find a solution $2\pi/k$-periodic with respect to x, satisfying the the additional condition

$$\int\limits_0^{2\pi/k}\left(\frac{1}{2}\mu_0^2\sin\chi\zeta_m + \int\limits_0^{\mu_0} u_m dy + Q_m|_{y=\mu_0}\right)dx = 0, \tag{1.10}$$

flowing out of the condition for the mass flow rate (1.6) and serving, as can be seen from what follows, for determination of the mean thickness of the film. Here F_m, T_m, L_m, S_m, and Q_m are known inhomogeneities depending on quantities with an index less than the number m. Specifically,

$$F_1 = G_1 = K_1 = L_1 = S_1 = Q_1 = 0,\ F_2 = u_1 v_{1x} - v_1 u_{1x},$$
$$G_2 = u_1 u_{1x} - v_1 v_{1x} + v_1 \tau_1 - c_1 u_{1x},$$
$$K_2 = (\zeta_1 u_1)_x - c_1 \zeta_{1x},\ L_2 = -\zeta_1 D\sigma_1,$$
$$S_2 = 4\zeta_{1x} u_{1x} - \zeta_1 D\tau_1,\ Q_2 = \zeta_1 u_1,$$
$$F_3 = u_1 v_{2x} + u_2 v_{1x} - v_1 u_{2x} - v_2 u_{1x} - c_2 v_{1x} - c_1 v_{2x} + \delta\mu_2 \sin\chi y v_{1x},$$
$$G_3 = (u_1 u_2 - v_1 v_2)_x + v_1 \tau_2 + v_2 \tau_1 - c_2 u_{1x} - c_1 u_{2x} + \delta\mu_2 \sin\chi (y u_{1x} + v_1),$$
$$K_3 = \frac{\partial}{\partial x}\Big[\delta\mu_2(u_1 + \mu_0 \sin\chi \zeta_1) + \zeta_1 u_2 + \zeta_2 u_1 + \frac{1}{2}\zeta_1^2 D u_1 - $$
$$- \frac{1}{6}\sin\chi \zeta_1^3 - c_2\zeta_1 - c_1\zeta_2\Big],$$
$$L_3 = 4u_{1x}\zeta_{1x}^2 - \frac{3}{2}\gamma\zeta_{1xx}\zeta_{1x}^2 - \zeta_2 D\sigma_1 - \zeta_1 D\sigma_2 - \delta\mu_2 D\sigma_1 - \frac{1}{2}\zeta_1^2 D^2\sigma_1,$$
$$S_3 = 4(u_{1x}\zeta_{2x} + u_{2x}\zeta_{1x} + \zeta_1\zeta_{1x} D u_{1x}) - \delta\mu_2 D\tau_1 - \zeta_2 D\tau_1 - \zeta_1 D\tau_2,$$
$$Q_3 = \zeta_2 u_1 + \zeta_1 u_2 + \frac{1}{2}\zeta_1^2 D u_1 - \frac{1}{6}\zeta_1^3 \sin\chi.$$

For m = 1, we obtain a linear homogeneous problem for calculation of the eigenvector and the critical values of the parameters Re_0 and c_0. We seek its solution in the form

$$\begin{pmatrix} u_1 \\ v_1 \\ \sigma_1 \\ \tau_1 \\ \zeta_1 \end{pmatrix} = \beta \left(e^{ikx} \begin{pmatrix} u_{1,1}(y) \\ v_{1,1}(y) \\ \sigma_{1,1}(y) \\ \tau_{1,1}(y) \\ \zeta_{1,1} \end{pmatrix} + e^{-ikx} \begin{pmatrix} \bar{u}_{1,1}(y) \\ \bar{v}_{1,1}(y) \\ \bar{\sigma}_{1,1}(y) \\ \bar{\tau}_{1,1}(y) \\ \bar{\zeta}_{1,1} \end{pmatrix} \right),$$

where β is a constant, subject to determination, which, without loss of generality, can be assumed to be positive (in the contrary case, it would be necessary to shift the origin of the reckoning $x \to x + \pi/k$). An overscore denotes complex conjugation. As a normalizing condition it is convenient to take $\zeta_{1,1} = 1$. For such a choice of $\zeta_1 = 2\beta \cos kx$ and, consequently, for small values of ε, the quantity 2βε can be interpreted as the amplitude of the waves at the free surface of the liquid. After separation of the variable x we arrive at the ordinary differential equations

$$Du_{1,1} = \tau_{1,1} - ikv_{1,1},\ Dv_{1,1} = -iku_{1,1},$$
$$D\sigma_{1,1} = ik[(U - c_0)v_{1,1} - \tau_{1,1}], \tag{1.11}$$
$$D\tau_{1,1} = ik[(U - c_0)u_{1,1} - \sigma_{1,1}] + 4k^2 u_{1,1} + \mathrm{DU} v_{1,1}$$

with the boundary conditions

$$u_{1,1} = v_{1,1} = 0\ (y = 0); \tag{1.12}$$
$$\sigma_{1,1} = -\cos\chi - \gamma k^2,\ \tau_{1,1} = \sin\chi\ (y = \mu_0); \tag{1.13}$$
$$v_{1,1} = ikV\ (y = \mu_0). \tag{1.14}$$

To construct the conjugate problem, for m = 1 we multiply the first equation of system (1.9) by the function $\bar{\Lambda}(x, y)$, the second by $\bar{\Theta}(x, y)$, the third by $\bar{\Phi}(x, y)$, and the fourth by $\bar{\Psi}(x, y)$, and we integrate around the rectangle $\{0 \leqslant x \leqslant 2\pi/k,\ 0 \leqslant y \leqslant \mu_0\}$, using the periodicity with respect to x (period 2π/k) and the conditions at y = 0, μ_0 for the quantities u_1, v_1, σ_1, and τ_1. We find the boundary conditions for the functions introduced into the discussion, requiring the reversion to zero of the terms outside the integral signs, arising with integration by parts. As a result, we arrive at the conjugated problem

$$D\Lambda = 4\Psi_{xx} + (U - c_0)\Psi_x - \Theta_x,\quad D\Theta = (U - c_0)\Phi_x - \mathrm{DU}\cdot\Psi - \Lambda_x,$$
$$D\Phi = -\Psi_x,\ D\Psi = -\Lambda - \Phi_x,\ \Phi = \Psi = 0\ (y = 0),\ \Lambda = 0\ (y = \mu_0),$$
$$\sin\chi\cdot\Psi - V\Theta_x + \gamma\Phi_{xx} - \cos\chi\cdot\Phi = 0\ (y = \mu_0),$$

which, after separation of the variable x

$$\{\Lambda, \Theta, \Phi, \Psi\} = e^{ikx}\{\overline{\lambda}(y), \overline{\theta}(y), \overline{\varphi}(y), \overline{\psi}(y)\}$$

and the introductionof the normalizing factor $\theta = 1$ for $y = \mu_0$, leads to the ordinary differential equations

$$D\lambda = ik[\theta - (U - c_0)\psi] - 4k^2\psi,$$
$$D\theta = ik[\lambda - (U - c_0)\varphi] - \mathrm{DU}\psi,\quad D\varphi = ik\psi,\ D\psi = ik\varphi - \lambda \tag{1.15}$$

with the boundary conditions

$$\varphi = \psi = 0\ (y = 0),\ \theta = 1\ (y = \mu_0); \tag{1.16}$$
$$(\cos\chi + \gamma k^2)\varphi - \sin\chi\psi = ikV\ (y = \mu_0); \tag{1.17}$$
$$\lambda = 0\ (y = \mu_0). \tag{1.18}$$

The condition of solvability of the inhomogeneous problem (1.9), having the form ($m = 2, 3, 4, \ldots$)

$$\int_0^{\mu_0}\int_0^{2\pi/k} (F_m\varphi + G_m\psi)\,e^{-ikx}dxdy = \int_0^{2\pi/k} (K_m\theta + L_m\varphi + S_m\psi)\,|_{y=\mu_0}e^{-ikx}dx, \tag{1.19}$$

for $m = 2$ permits the conclusion that $c_1 = 0$ if the value of

$$I_1 = \theta(\mu_0) - \int_0^{\mu_0} (v_{1,1}\varphi + u_{1,1}\psi)\,dy$$

differs from zero. The latter inequality was verified numerically and, in the case under consideration, was found to be satisfied. The solution of the problem (1.9) (1.10) for $m = 2$ is given by the formulas

$$u_2 = \beta^2[y\zeta_{2,0}\sin\chi + u_{2,0}(y) + u_{2,2}(y)e^{2ikx} + \overline{u}_{2,2}(y)e^{-2ikx}],$$
$$v_2 = \beta^2[v_{2,2}(y)e^{2ikx} + \overline{v}_{2,2}(y)e^{-2ikx}],$$
$$\sigma_2 = \beta^2[-\zeta_{2,0}\cos\chi + \sigma_{2,0}(y) + \sigma_{2,2}(y)e^{2ikx} + \overline{\sigma}_{2,2}(y)e^{-2ikx}],$$
$$\tau_2 = \beta^2[\zeta_{2,0}\sin\chi + \tau_{2,0}(y) + \tau_{2,2}(y)e^{2ikx} + \overline{\tau}_{2,2}(y)e^{-2ikx}]$$
$$\zeta_2 = \beta^2[\zeta_{2,0} + \zeta_{2,2}e^{2ikx} + \overline{\zeta}_{2,2}e^{-2ikx}],$$

here the constant $\zeta_{2,0}$, introduced above, is uniquely determined using the condition (1.10) and is found to be

$$\zeta_{2,0} = -\frac{1}{\mu_0^2\sin\chi}\left[\int_0^{\mu_0} u_{2,0}(y)\,dy + 2\mathrm{Real}\,u_{1,1}(\mu_0)\right].$$

We find the remaining values by solving the boundary-value problems

$$Du_{2,0} = \tau_{2,0},\ u_{2,0} = 0\ (y = 0),$$
$$D\sigma_{2,0} = 4k\,\mathrm{Im}\,(u_{1,1}\overline{v}_{1,1}),\quad D\tau_{2,0} = 2\,\mathrm{Real}\,(v_{1,1}\overline{\tau}_{1,1}),$$
$$\sigma_{2,0} = 2k^2V^2\ (y = \mu_0),\ \tau_{2,0} = 2kV\,\mathrm{Im}u_{1,1}\ (y = \mu_0); \tag{1.20}$$
$$Du_{2,2} = \tau_{2,2} - 2ikv_{2,2},\ Dv_{2,2} = -2iku_{2,2},$$
$$D\sigma_{2,2} = 2ik[(U - c_0)v_{2,2} - \tau_{2,2}],\ D\zeta_{2,2} = 0,$$
$$D\tau_{2,2} = 2ik\,[(U - c_0)\,u_{2,2} - \sigma_{2,2}] + 16k^2u_{2,2} + \mathrm{DU}\,v_{2,2} + v_{1,1}\tau_{1,1} + ik\,(u_{1,1}^2 - v_{1,1}^2), \tag{1.21}$$

$$u_{2,2} = v_{2,2} = 0 \ (y = 0), \ v_{2,2} - 2ikV\zeta_{2,2} = 2iku_{1,1} \ (y = \mu_0), \tag{1.21}$$
$$\sigma_{2,2} + (\cos\chi + 4\gamma k^2)\zeta_{2,2} = k^2V^2 + ik\sin\chi \ (y = \mu_0),$$
$$\tau_{2,2} - \zeta_{2,2}\sin\chi = -(8k^2 + ikV)u_{1,1} - ik(\cos\chi + \gamma k^2) \ (y = \mu_0).$$

Then, substituting m = 3 into the conditions of solvability (1.19), we arrive at a complex equation for determining the two real constants β and c_2:

$$ikc_2I_1 + \beta^2I_2 = \delta\mu_2I_3$$

solving this, we obtain

$$\beta = \sqrt{\delta\mu_2\frac{\mathrm{Real}\,(I_3\bar{I}_1)}{\mathrm{Real}\,(I_1\bar{I}_2)}}, \quad c_2 = \frac{\delta\mu_2\,\mathrm{Im}\,(I_3\bar{I}_2)}{k\,\mathrm{Real}\,(I_1\bar{I}_2)}. \tag{1.22}$$

Here the sign of δ is so selected that the expression under the radical sign will be nonnegative; the coefficients I_2 and I_3 are calculated using the following formulas:

$$I_2 = I_4 - \zeta_{2,0}I_3, \ I_3 = -\sin\chi\int_0^{\mu_0}[iky\,(v_{1,1}\varphi + u_{1,1}\psi) + v_{1,1}\psi]\,dy +$$
$$+ [ik\,(u_{1,1} - \mu_0\sin\chi)\,\theta - \varphi D\sigma_{1,1} - \psi D\tau_{1,1}]_{y=\mu_0},$$
$$I_4 = \int_0^{\mu_0}(ik\varphi z_1 + \psi z_2)\,dy - (ik\theta z_3 + \varphi z_4 + \psi z_5)\,|_{y=\mu_0},$$
$$z_1 = v_{1,1}u_{2,0} + 3(\bar{u}_{1,1}v_{2,2} - u_{2,2}\bar{v}_{1,1}),$$
$$z_2 = v_{1,1}\tau_{2,0} + \bar{v}_{1,1}\tau_{2,2} + v_{2,2}\bar{\tau}_{1,1} + ik(u_{1,1}u_{2,0} + \bar{u}_{1,1}u_{2,2} - \bar{v}_{1,1}v_{2,2}),$$
$$z_3 = u_{2,0} + u_{2,2} + \zeta_{2,2}\bar{u}_{1,1} + Du_{1,1} + 0{,}5(D\bar{u}_{1,1} - \sin\chi),$$
$$z_4 = 1{,}5\gamma k^4 + 4ik^3(\bar{u}_{1,1} + 2u_{1,1}) - \zeta_{2,2}D\bar{\sigma}_{1,1} - D\sigma_{2,0} - D\sigma_{2,2} - D^2\sigma_{1,1} - 0{,}5D^2\bar{\sigma}_{1,1},$$
$$z_5 = 8k^2(\zeta_{2,2}\bar{u}_{1,1} + u_{2,2} + 0{,}5D\bar{u}_{1,1}) - \zeta_{2,2}D\bar{\tau}_{1,1} - D\tau_{2,0} - D\tau_{2,2} - D^2\tau_{1,1} - 0{,}5D^2\bar{\tau}_{1,1}.$$

As can be seen from (1.22), the values of the constants β and c_2 are determined and β differs from zero if the following conditions are satisfied:

$$\mathrm{Real}(I_3\bar{I}_1) \neq 0, \ \mathrm{Real}(I_1\bar{I}_2) \neq 0, \tag{1.23}$$

from which we obtain [22, 23] the convergence of the series (1.8) and the singularity (with an accuracy to the shift $x \to x + \mathrm{const}$) for small values of ε of the autovibrational conditions (1.8), responsible for the plane-parallel flow (1.7) and existing in the supercritical region $\mathrm{Re} > \mathrm{Re}_0$ for $\delta = +1$ or in the subcritical region $\mathrm{Re} < \mathrm{Re}_0$ in the case $\delta = -1$. The autovibrations have the form of nonlinear waves, running downward along the flow as a result of the positive nature of the velocity c_0.

The above-described method was used on an ODRA-1204 computer to make two series of calculations of secondary wavy flows near the threshold of stability for fixed values of the parameters χ and γ and different wave numbers: $\chi = 45°$, $\gamma = 3387$; 2) $\chi = 90°$, $\gamma = 2903$. The constant γ for the first series of calculations was obtained for water at 20°C ($\rho = 0.9982$ g/cm³, $\nu = 1.004 \cdot 10^{-2}$ cm²/sec, T = 72.75 dyn/cm); the values of χ and γ for the second series correspond to the conditions of the experiment of [2] (water 15°C, $\rho = 1$ g/cm³, $\nu = 1.14 \cdot 10^{-2}$ cm²/sec, T = 74 dyn/cm). Here an investigation was m de of the character of the branching of the solutions of the equations of hydrodynamics, both for perturbations of the type of surface waves [9] and for shear waves. Previously, numerical integration of Eqs. (1.11) was used to find the critical values of the phase velocity c_0 and the Reynolds number Re_0 with a high degree of exactness; for large values of Re_0, due to the rapid growth and the oscillations of the solutions of the differential equations the method of differential successive fitting was used [25]. The results obtained for the case of surface waves are illustrated in Fig. 1. The numbers 1 and 2 on the curves indicate that the curve was plotted for the set of parameters 1 or 2. The stability limit corresponding to the appearance of Tollmein—Schlichting shear waves is attained at considerably greater Reynolds numbers. The central curve, which in this case has the form of a tongue, is shown in Fig. 2 ($\chi = 45°$, $\gamma = 3387$).

We note the existence of the vertical asymptote $k = k_*$ ($k_* \approx 0.102$ in case 1 and $k_* \approx 0.121$ in case 2) on the curve of the dependence $\mathrm{Re}_0(k)$ for a mode corresponding to surface waves; the latter exist only for $k < k_*$ and are exponentially damped for $k \geq k_*$ as a result of the stabilizing action of the surface tension. The critical Reynolds numbers $\mathrm{Re}_0(k)$ rise

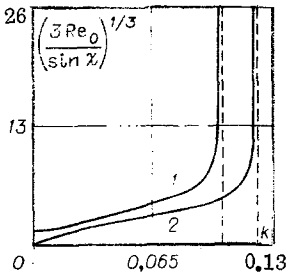

Fig. 1

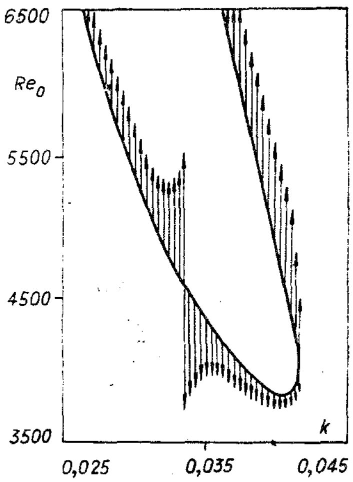

Fig. 2

unboundedly as $k \to k_*$, and the phase velocity of the waves approaches the velocity of the unperturbed parabolic flow at the free boundary.

We note that the calculated neutral curves differ from the analogous curves of [9], where different scales of length, time, and determining parameters were selected: The calculations in [9] were made for fixed values of the parameter $W = \gamma\mu_0$, and a dimensionless number $\alpha = k\mu$) — based on the thickness of the layer was used.

After finding the eigennumbers, the boundary-value problems (1.11)-(1.13), (1.15)-(1.17) (1.20), and (1.21) were solved consecutively using a complex variant of the method of orthogonalization [26] based on an ALGOL program developed in [27, 28], the functionals $\zeta_{2,0}$, I_1, I_2, and I_3 were calculated, and the coefficients δ, β, and c_2 were found. Here the calculation of the integrals entering into the functionals was reduced to the solution of the Cauchy problem with a zero initial condition at $y = 0$ and was carried out simultaneously with the numerical integration of the system of differential equations by the Runge–Kutta method. The "superfluous" boundary conditions (1.14) and (1.18) were not used in the solution of the boundary-value problems; the exactness of the satisfaction of the discarded boundary conditions was determined by the exactness of the assignment of the eigenvalues of Re_0 and c_0: For ideally exact values of Re_0 and c_0 and ideally exact integration, the conditions (1.14) and (1.18) should automatically be satisfied with absolute exactness. This fact was used for purposes of control.

Some of the numerical results obtained are given in Table 1. It was found that in the case of surface waves, for all wave numbers in the range $0 < k < k_*$, secondary conditions exist only in the supercritical region $Re > Re_0$. The most clearly marked special characteristics of secondary flow, having the form of surface waves, appear with large Reynolds numbers. Curves of some of the components of the solution for this case are given in Figs. 3 and 4 ($\chi = 45°$, $\gamma = 3387$, $k = 0.1019$, $Re_0 = 2900$, $c_0 = 200.5$, $\beta = 8.40 \cdot 10^{-4}$, $c_2 = 4.33 \cdot 10^{-2}$, $\zeta_{2,0} = -1.41 \cdot 10^{-2}$, $\zeta_{2,2} = -9.40 \cdot 10^{-2} - 3.91 \cdot 10^{-3} i$); Fig. 3: curve 1) Real $u_{1,1}$, 2) 100 Im$u_{1,1}$, 3) 100 Real $v_{1,1}$, 4) 10 Im $v_{1,1}$; Fig. 4: curve 1) 5 $u_{2,0}$, 2) Real $u_{2,2}$, 3) 10 Im $u_{2,2}$.

Calculations showed that the character of the branching of steady-state flow (1.7) with the formation of shear waves is determined by the value of the wave number k. The values of the constant β found are shown in Fig. 2 in the form of arrows ($\chi = 45°$, $\gamma = 3387$); the arrows are directed upward if the secondary wavy conditions are branched for $Re > Re_0$ and downward in the contrary case. The length of an arrow characterizes the numerical value of the constant β, reverting, respectively, to infinity (zero) at the left-hand (right-hand) end of the interval of wave numbers for which the branching is subcritical (see Fig. 2). The reversion of β to zero at the extreme right-hand point $k = k_{max}$ is due to the coalescence of the upper and lower branches of the neutral curve ($\beta \sim \text{const}\sqrt{k_{max} - k}$ as $k \to k_{max} - 0$); with an approach to the left-hand end $k = k_0$ of the interval of wave numbers, the denominator Real $(I_1\bar{I}_2)$ tends toward zero, so that the value of β rises unboundedly: $\beta \sim \text{const}\,|k - k_0|^{-1/2}$ $(k \to k_0)$. At these two exceptional points, $k = k_0$ and $k = k_{max}$, conditions (1.23) are not satisfied and the expansions (1.8) lose their force.

TABLE 1

k	Re_0	c_0	β	c_2	$\zeta_{2,0}$	δ	Note
0,03	4,525	5,037	0,1980	0,5063	—0,7385	+1	Surface waves
0,06	18,41	12,23	0,2351	0,1524	—0,3892	+1	$\chi=45°$, $\gamma=3387$
0,08	40,35	19,28	0,2224	$0,9548\cdot10^{-1}$	—0,1839	+1	
0,10	275,0	50,82	$0,6609\cdot10^{-1}$	$0,8990\cdot10^{-1}$	$-0,5609\cdot10^{-1}$	+1	
$2,802\cdot10^{-2}$	6000	67,33	$0,250\cdot10^{-1}$	$0,223\cdot10^{-1}$	0,694	+1	Shear waves
$3,260\cdot10^{-2}$	4800	62,64	$0,519\cdot10^{-1}$	$0,775\cdot10^{-1}$	0,741	+1	$\chi=45°$, $\gamma=3387$
$3,693\cdot10^{-2}$	4100	59,98	$0,217\cdot10^{-1}$	$0,366\cdot10^{-2}$	0,810	—1	
$4,044\cdot10^{-2}$	3820*	59,84	$0,898\cdot10^{-2}$	$-0,749\cdot10^{-2}$	0,924	—1	
$4,014\cdot10^{-2}$	4792	68,83	$0,532\cdot10^{-2}$	$0,104\cdot10^{-1}$	$0,128\cdot10^{1}$	+1	
$3,739\cdot10^{-2}$	5992	76,87	$0,557\cdot10^{-2}$	$0,986\cdot10^{-2}$	$0,150\cdot10^{1}$	+1	
0,036	3,199	4,490	0,1594	0,7312	—0,9318	+1	Surface waves
0,070	14,23	11,59	0,2165	0,1941	—0,4969	+1	$\chi=90°$, $\gamma=2903$
0,115	110,3	34,87	0,1183	0,1163	$-0,9862\cdot10^{-1}$	+1	

Note. An asterisk denotes a minimal value of the Reynolds number at a neutral curve corresponding to shear waves.

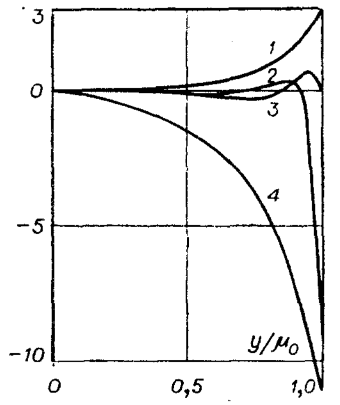

Fig. 3

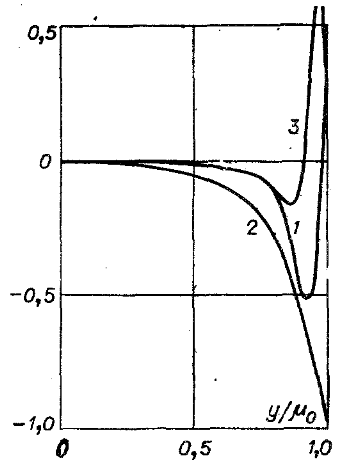

Fig. 4

2. Waves of Finite Amplitude

The method used in Sec. 1 for calculating nonlinear waves of small amplitude must be limited to some neighborhood of the neutral curve and the question of the applicability of the solution found, containing two terms of the expansion, with concrete numerical values of the parameter ε, remains open. The direct numerical method proposed below for calculating surface waves reduces the problem to a system of nonlinear algebraic equations and is found to be applicable up to Reynolds numbers several times greater than the critical.

We shall study the motion of a liquid in a movable system of coordinate Oxy, moving along an inclined plane with the velocity c, equal to the phase velocity of a wave. In this system of reckoning the wavy flow becomes fully established. We write the equations of motion in the Gromeko—Lamb form

$$\begin{gathered}\partial U/\partial y = V_x - \Omega,\ \partial V/\partial y = -U_x,\\ \partial \Omega/\partial y = \sin\chi + \Omega V - H_x,\\ \partial H/\partial y = -\cos\chi - \Omega U + \Omega_x,\end{gathered} \tag{2.1}$$

taking the following as the dependent variables: U(x, y) is the longitudinal component of the velocity (referred to the system of reckoning Oxy); V(x, y) is the transverse component of the velocity vector; Ω(x, y) is a vortex; and $H(x, y) = p - p_a + (U^2 + V^2)/2$ is the total pressure, reckoned from the level p_a. These quantities, together with the function $\zeta(x)$ describing the form of the free surface, are periodic with respect to x with a given period $2\pi/k$ and satisfy the conditions

$$U = -c,\ V = 0\ (y = 0),\ \tau = 2V_x - \Omega; \tag{2.2}$$

$$\mathrm{Re} - \int_0^{\zeta} U dy - \frac{ck}{2\pi}\int_0^{2\pi/k} \zeta dx = 0; \tag{2.3}$$

$$(\zeta_x^2 - 1)\tau + 4\zeta_x U_x = 0 \qquad (y = \zeta); \tag{2.4}$$

$$H - 0.5(U^2 + V^2) + 2U_x + \tau\zeta_x + \gamma\zeta_{xx}(1 + \zeta_x^2)^{-3/2} = 0 \qquad (y = \zeta), \tag{2.5}$$

equivalent to the relationships (1.2)-(1.6).

We bring the problem to a system of nonlinear algebraic equations. To this end, we expand the functions U, V, Ω, and H in power series in terms of the transverse coordinate y,

$$\{U, V, \Omega, H\} = \sum_{m=0}^{M} \{U_m(x), V_m(x), \Omega_m(x), H_m(x)\} y^m, \tag{2.6}$$

limiting ourselves to a finite number of terms, and then we use an expansion in a Fourier series,

$$\{U_m(x),\ V_m(x),\ \Omega_m(x),\ H_m(x),\ \zeta(x)\} = \sum_{n=-N}^{N} \{U_{m,n},\ V_{m,n},\ \Omega_{m,n},\ H_{m,n},\ \eta_n\}\, e^{inkx}, \tag{2.7}$$

discarding all harmonics with a number $|n| > N$ and simultaneously setting $\mathrm{Im}\eta_1 = 0$. The latter requirement makes it possible to eliminate arbitrary shifts of the start of the reckoning at the x axis. We note that Fourier coefficients with a negative index $-n$ are obtained from coefficients with the index n by an operation of complex conjugation. This guarantees the real nature of the sum in (2.7). As the unknowns of the sought nonlinear algebraic system we take the velocity of the wave c and the numbers $\eta_n, \Omega_{0,n}$, and $H_{0,n}$ ($n = 0, 1, 2, \ldots, N$), which, taking account of the real nature of η_0, η_1, $\Omega_{0,0}$, and $H_{0,0}$, gives $6N + 3$ unknowns in real form. The equations for their determination give the conditions (2.3)-(2.5). In actual fact, the equations of motions (2.1), after the substitutions (2.6) and (2.7), reduce to a recurrent system with respect to the numerical coefficients,

$$\begin{gathered}(m+1)U_{m+1,n} = iknV_{m,n} - \Omega_{m,n},\ (m+1)V_{m+1,n} = -iknU_{m,n},\\ (m+1)\,\Omega_{m+1,n} = \sin\chi\delta_{m,n} - iknH_{m,n} + \sum_{s=0}^{m} \langle \Omega_s(x)\, V_{m-s}(x)\rangle_n,\\ (m+1)\,H_{m+1,n} = -\cos\chi\delta_{m,n} + ikn\Omega_{m,n} - \sum_{s=0}^{m} \langle \Omega_s(x)\, U_{m-s}(x)\rangle_n,\\ U_{0,n} = -c\delta_{0,n},\ V_{0,n} = 0\ (n = 0,\ 1, \ldots, N;\ m = 0,\ 1, \ldots, M-1),\end{gathered} \tag{2.8}$$

which, for designated values of c, $\Omega_{0,n}$, and $H_{0,n}$ ($n = 0, 1, \ldots, N$), makes it possible to consecutively calculate all the coefficients $U_{m,n}$, $V_{m,n}$, $\Omega_{m,n}$, and $H_{m,n}$ with an index $m > 0$. The value of $\delta_{m,n}$ figuring in (2.8) is assumed equal to unity if $m = n = 0$ and equal to zero in the contrary case; the symbol $\langle\ \rangle_n$ denotes a Fourier coefficient with the harmonic exp(inkx); it can be shown that if the functions $a(x)$ and $b(x)$ are segments of the Fourier series

$$a(x) = \sum_{n=-N_a}^{N_a} a_n e^{inkx},\quad b(x) = \sum_{n=-N_b}^{N_b} b_n e^{inkx},$$

then the following formula holds:

$$\langle a(x)\, b(x)\rangle_n = \begin{cases} \displaystyle\sum_{s=-\min(N_a, N_b-n)}^{\min(N_a, N_b+n)} a_s b_{n-s}, & |n| \leqslant N_a + N_b,\\ 0, & |n| > N_a + N_b,\end{cases}$$

which, using a computer, makes it possible to find the coefficients of the quantities U, V, U_x, V_x, etc., in a Fourier series at the free boundary $y = \zeta(x)$, using (2.6) and the Horner algorithm for calculating the values of the polynomial. The Fourier coefficients for the curvature $C(x) = \zeta_{xx}(1 + \zeta_x^2)^{-3/2}$ can be calculated using the formulas of a harmonic analysis of the periodic function [29]:

$$\langle C(x)\rangle_n = \frac{1}{2(N+1)} \sum_{s=0}^{2N+1} C(x_s)\, e^{-inkx_s},\quad x_s = \frac{\pi s}{k(N+1)},\quad |n| \leqslant N.$$

Then, substituting the Fourier expansions found into the left-hand sides of the equalities (2.3)-(2.5), we perform the indicated multiplication of the Fourier series, collect coefficients with the harmonics exp(inkx) ($n = 0, 1, 2, \ldots, N$), and equate them to zero in accordance with the kind of the right-hand-sides. As a result, this gives in real form a system of $6N + 3$ nonlinear equations with respect to such a number of unknowns.

Fictitious calculations of surface waves were made for $\chi = 90°$ and different Reynolds numbers for values of γ corresponding to the experiments of [2] with water ($\gamma = 2903$) and alcohol ($\gamma = 530.5$, $\rho = 0.79$ g/cm³, $\nu = 2.02\cdot10^{-2}$ cm²/sec, and $T = 22.9$ dyn/cm). Here, in the final series of calculations, it was assumed that $M = 10$ and $N = 5$; the system of 33 nonlinear algebraic equations was solved by the Newton method, with approximation of the partial derivatives entering into the Jacobian by finite differences. The dimensionless wave number was given on the basis of experimental data as equal to 0.036 for water and 0.062 for alcohol. Some of the numerical results obtained are given in Table 2. It was found that the mean thickness of the film η_0 for the wavy downflow of a liquid is less than the thickness of the layer for plane-parallel flow (1.7) at the same Reynolds number (the difference $\mu - \eta_0$ is positive, see Table 2). This circumstance has been noted in a number of experiments [30].

TABLE 2

Re	c	$\mu-\eta_0$	η_1	η_2	η_3	η_4	η_5	Note
3,58	4,75	$8,96\cdot10^{-3}$	0,1	$-2,96\cdot10^{-3}$ $-8,05\cdot10^{-[illegible]}i$	$-1,90\cdot10^{-4}$ $+5,56\cdot10^{-4}i$	$3,76\cdot10^{-5}$ $-2,90\cdot10^{-5}i$	$-4,16\cdot10^{-6}$ $+4,66\cdot10^{-7}i$	
4,61	5,39	$3,27\cdot10^{-2}$	0,2	$-1,15\cdot10^{-2}$ $-3,02\cdot10^{-2}i$	$-1,25\cdot10^{-3}$ $+4,04\cdot10^{-3}i$	$4,96\cdot10^{-4}$ $-4,14\cdot10^{-4}i$	$-1,02\cdot10^{-4}$ $+1,65\cdot10^{-5}i$	Water
6,26	6,20	$6,60\cdot10^{-2}$	0,3	$-2,61\cdot10^{-2}$ $-6,37\cdot10^{-2}i$	$-3,29\cdot10^{-3}$ $+1,26\cdot10^{-2}i$	$2,20\cdot10^{-3}$ $-2,04\cdot10^{-3}i$	$-6,17\cdot10^{-4}$ $-7,84\cdot10^{-5}i$	$k=0,036$
7,35	6,65	$8,50\cdot10^{-2}$	0,35	$-3,54\cdot10^{-2}$ $-8,41\cdot10^{-2}i$	$-3,40\cdot10^{-3}$ $+1,87\cdot10^{-2}i$	$4,25\cdot10^{-3}$ $-5,14\cdot10^{-3}i$	$-1,45\cdot10^{-3}$ $-1,47\cdot10^{-3}i$	$\gamma=2903$
2,20	3,40	$1,04\cdot10^{-2}$	0,1	$-2,77\cdot10^{-3}$ $-1,25\cdot10^{-2}i$	$-6,65\cdot10^{-4}$ $+9,58\cdot10^{-4}i$	$1,18\cdot10^{-4}$ $-3,26\cdot10^{-5}i$	$-1,29\cdot10^{-5}$ $-5,16\cdot10^{-6}i$	Alcohol
3,30	4,19	$3,65\cdot10^{-2}$	0,2	$-1,04\cdot10^{-2}$ $-4,55\cdot10^{-2}i$	$-4,24\cdot10^{-3}$ $+6,63\cdot10^{-3}i$	$1,46\cdot10^{-3}$ $-4,62\cdot10^{-4}i$	$-2,78\cdot10^{-4}$ $-1,08\cdot10^{-4}i$	$k=0,062$
5,07	5,13	$7,12\cdot10^{-2}$	0,3	$-2,27\cdot10^{-2}$ $-9,21\cdot10^{-2}i$	$-6,51\cdot10^{-3}$ $+2,17\cdot10^{-2}i$	$1,02\cdot10^{-2}$ $-4,86\cdot10^{-3}i$	$1,13\cdot10^{-4}$ $-4,05\cdot10^{-3}i$	$\gamma=530,5$

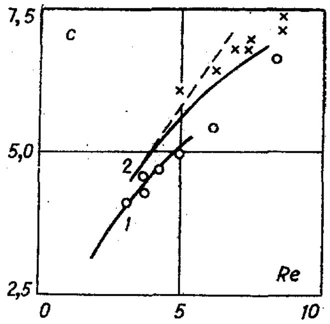

Fig. 5

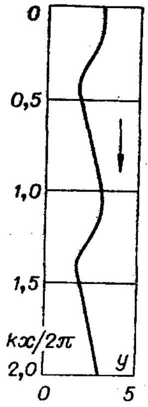

Fig. 6

The calculated dependence of the phase velocity on the Reynolds number for alcohol (solid line 1) and for water (solid line 2) is shown together with the experimental data of [2] in Fig. 5 (the circles represent experiments with alcohol and the crosses represent experiments with water); the dashed line is a curve of the dependence $c = c_0 + c_2(Re - Re_0)$, plotted for the case $\gamma = 2903$ and $k = 0.036$ on the basis of the calculations of Sec. 1. A characteristic profile of a nonlinear wave at a free surface is shown in Fig. 6 ($\gamma = 530.5$, $k = 0.062$, $M = 10$, $N = 5$, $Re = 5.07$, $c = 5.13$); the direction of the flow of liquid is shown by the arrow.

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STABILIZATION OF CONVECTIVE FLOW IN A VERTICAL LAYER USING A PERMEABLE PARTITION

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UDC 536.25

INTRODUCTION

The control of the stability of convective motions is one of the problems of applied hydrodynamics, since a loss of stability leads to a lowering to the characteristics of a number of technical objects (thermodiffusion columns, vertical heat-insulating layers, etc.). Some methods for the stabilization of convective flows are discussed in [1].

In the present article an investigation is made of the effect of a thin permeable partition, located at the interface between counterflows, on the stability of convective flow. A special characteristic of this means of stabilization is that a permeable partition, preventing the development of secondary motions, in practice changes the profile of steady-state flow and processes of molcular transfer. The effect of a permeable partition on the stability of a horizontal layer of liquid heated from below and of isothermal flow with a cubic velocity profile was investigated earlier in [2, 3].

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